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Concise formulary

Some definitions and formulae from the main text are represented in concise form[1].

1 Q0-triangle

Let be j, m, n natural numbers, k an integer number with |(n+k)/2| smaller or equal n, i.e. k is contained in the interval [-n,n], starting with -n, stepping by 2 until +n . Furthermore p is a number contained in the interval [0,1]. We define

$$Q0P(n,k,p) := \frac{p^{(n+k)/2}\,(1-p)^{(n-k)/2}\,n!}{\left(\frac{n+k}{2}\right)!\,\left(\frac{n-k}{2}\right)!}$$

The function Q0P(n,k,p) represents the probability of reaching coordinate k after n steps of a Bernoulli random walk, if the probability of a step in positive k-direction (e.g. right hand direction) is equal to p (and so for a step to the opposite direction is equal to 1-p). The numbers Q0 (n, k) of the Q0-triangle correspond to the special case of same probabilities for steps to the right and to the left, i.e. for p=(1-p)=0.5:

$$Q0(n,k) = Q0P(n,k,0.5)$$

The probabilities Q0Z (n) for return ("central meeting probabilities") correspond to the special case k = 0:

$$Q0Z(n) = Q0P(n,0,0.5) = \frac{n!}{2^n\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}$$

2 Q1-triangle

The Q1-triangle results from a superposition of two Q0-triangles with opposite sign, starting in position n=1, k=±1 after multiplication by 1/2. Addition of both means a "discrete differentiation" along k.

$$Q1(n,k) = -\frac{k}{n}\,Q0(n,k)$$

The absolute values |Q1 (n, k)| also arise, if after starting in row n=1 the following rows are constructed in usual way, but the numbers in the row centers k=0 are set to 0 in every row with even row number respectively, are so to speak "flown out", so that they can't be sources subsequently. Let for every even row number n be -Q2Z(n) "flowing out probability" there, i.e. the probability for flowing out centrally. Q2Z(n) is equivalent to the 1nd (discrete) derivative of Q1(n,k) in k=0 along k, i.e. Q2Z(n) = (Q1(n-1,1)-Q1(n-1,-1))/2; so Q2Z(n) is in k=0 the 2nd derivative of Q0(n,k) along k. It holds:

$$Q2Z(n) = \frac{Q1(n-1,1) - Q1(n-1,-1)}{2} = -\frac{n!}{2^n(n-1)\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!} = -\frac{Q0Z(n)}{n-1}$$

3 Q0M-triangle

[1] In wqm (contained in the download of the older texts) is a more extensive formulary.

$$Q0M(n,k) := \frac{(-0.5)^{(n+k)/2}\ 0.5^{\ (n-k)/2}\ n!}{\left(\frac{n+k}{2}\right)!\ \left(\frac{n-k}{2}\right)!} = (-1)^{(n+k)/2}\ Q0\ (n,k)$$

```
Q0M(n,k) is in case of  odd n antisymmetric and
           in case of even n     symmetric regarding to k=0 .
So addition behavior of right and left side is like the one of amplitudes of fermions resp. bosons.
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4 Taylor series expansions

$$\frac{1}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} Q0P\left(2n, 0, \frac{1+\sqrt{1-x^2}}{2}\right) = \sum_{n=0}^{\infty} Q0Z(2n)x^{2n}$$

$$\sqrt{1-x^2} = \sum_{n=0}^{\infty} Q2Z(2n)\ x^{2n}$$

$$\frac{1}{\sqrt{1+x^2}} = \sum_{n=0}^{\infty} Q0M(2n,0)\ x^{2n}$$

5 Limits

$$\lim_{n\to\infty}\left[\frac{\sum_{l=0}^{n}\sum_{m=0}^{l/2} Q0Z(2m)}{\sqrt{\frac{8n^3}{9\pi}}}, \frac{\sum_{m=0}^{n/2} Q0Z(2m)}{\sqrt{\frac{2n}{\pi}}}, \frac{Q0Z(n)}{\sqrt{\frac{2}{\pi\ n}}}, \frac{-Q2Z(n)}{\sqrt{\frac{2}{\pi\ n^3}}}\right] = [1,1,1,1]$$

6 Multiple discrete differentiation (Formation of higher-order finite differences)

Similarly to the analytical case multiple discrete differentiation can be defined recursively (by formation of higher-order finite differences). Let be QDP(d,n,k,p) the d times along k differentiated function Q0P (n, k, p), then

$$\text{QDP}(0,n,k,p) = Q0P(n,k,p)$$

and for n ≥ d ≥ 1

$$\text{QDP}(d,n,k,p) = \frac{\text{QDP}(d-1,n-1,k+1,p)-\text{QDP}(d-1,n-1,k-1,p)}{2}.$$

At this n ≥ d is necessary that enough values are available to build a (finite) difference of d-th order. Let DF(d, n, k) := QDP(d, n, k, 1/2) / Q0P(n, k, 1/2); We get

$$DF(1,n,k) = -\frac{\mathrm{k}}{\mathrm{n}}$$

$$DF(2,n,k) = \frac{\mathrm{k}^2 - \mathrm{n}}{n(n-1)}$$

$$DF(3,n,k) = \frac{-\mathrm{k}^3 + 3\mathrm{kn} - 2\mathrm{k}}{\mathrm{n}(\mathrm{n}-1)(\mathrm{n}-2)}$$

$$DF(4,n,k) = \frac{\mathrm{k}^4 - 6\mathrm{k}^2\mathrm{n} + 8\mathrm{k}^2 + 3\mathrm{n}^2 - 6\mathrm{n}}{\mathrm{n}(\mathrm{n}-1)(\mathrm{n}-2)(\mathrm{n}-3)}$$

$$DF(5,n,k) = \frac{-k^5 + 10k^3n - 20k^3 - 15kn^2 + 50kn - 24k}{n(n-1)(n-2)(n-3)(n-4)}$$

$$DF(6,n,k) = \frac{k^6 - 15k^4n + 40k^4 + 45k^2n^2 - 210k^2n + 184k^2 - 15n^3 + 90n^2 - 120n}{n(n-1)(n-2)(n-3)(n-4)(n-5)}$$

1.1 Binomial coefficients and multiple differentiation (example matrix)

{BinCoeffDiffMatrix} The representation of operators as matrices is often useful in discrete considerations. Here a matrix representation of the operator for discrete differentiation in form of an example matrix with high number of dimensions for clarification of the development: Multiplication of a 15-dimensional vector by the following matrix

```
      ¦  0   1   0   0   0   0   0   0   0   0   0   0   0   0  -1 ¦
      ¦ -1   0   1   0   0   0   0   0   0   0   0   0   0   0   0 ¦
      ¦  0  -1   0   1   0   0   0   0   0   0   0   0   0   0   0 ¦
      ¦  0   0  -1   0   1   0   0   0   0   0   0   0   0   0   0 ¦
      ¦  0   0   0  -1   0   1   0   0   0   0   0   0   0   0   0 ¦
      ¦  0   0   0   0  -1   0   1   0   0   0   0   0   0   0   0 ¦
      ¦  0   0   0   0   0  -1   0   1   0   0   0   0   0   0   0 ¦
Δ  := ¦  0   0   0   0   0   0  -1   0   1   0   0   0   0   0   0 ¦ * 1/2
      ¦  0   0   0   0   0   0   0  -1   0   1   0   0   0   0   0 ¦
      ¦  0   0   0   0   0   0   0   0  -1   0   1   0   0   0   0 ¦
      ¦  0   0   0   0   0   0   0   0   0  -1   0   1   0   0   0 ¦
      ¦  0   0   0   0   0   0   0   0   0   0  -1   0   1   0   0 ¦
      ¦  0   0   0   0   0   0   0   0   0   0   0  -1   0   1   0 ¦
      ¦  0   0   0   0   0   0   0   0   0   0   0   0  -1   0   1 ¦
      ¦  1   0   0   0   0   0   0   0   0   0   0   0   0  -1   0 ¦
```

means first order discrete differentiation "along" the index k of the vector components (calculation of the finite first order difference - the shift dk of the index k is 2, therefore division by 2). Multiplication by the n-ten power Δ^n of this matrix yields n-th order discrete differentiation (formation of the finite n-th order difference). For example means multiplication by

```
      ¦ -20    0   15    0   -6    0    1    0    0    1    0   -6    0   15    0 ¦
      ¦   0  -20    0   15    0   -6    0    1    0    0    1    0   -6    0   15 ¦
      ¦  15    0  -20    0   15    0   -6    0    1    0    0    1    0   -6    0 ¦
      ¦   0   15    0  -20    0   15    0   -6    0    1    0    0    1    0   -6 ¦
      ¦  -6    0   15    0  -20    0   15    0   -6    0    1    0    0    1    0 ¦
      ¦   0   -6    0   15    0  -20    0   15    0   -6    0    1    0    0    1 ¦
 6    ¦   1    0   -6    0   15    0  -20    0   15    0   -6    0    1    0    0 ¦
Δ   = ¦   0    1    0   -6    0   15    0  -20    0   15    0   -6    0    1    0 ¦ * 1/64
      ¦   0    0    1    0   -6    0   15    0  -20    0   15    0   -6    0    1 ¦
      ¦   1    0    0    1    0   -6    0   15    0  -20    0   15    0   -6    0 ¦
      ¦   0    1    0    0    1    0   -6    0   15    0  -20    0   15    0   -6 ¦
      ¦  -6    0    1    0    0    1    0   -6    0   15    0  -20    0   15    0 ¦
      ¦   0   -6    0    1    0    0    1    0   -6    0   15    0  -20    0   15 ¦
      ¦  15    0   -6    0    1    0    0    1    0   -6    0   15    0  -20    0 ¦
      ¦   0   15    0   -6    0    1    0    0    1    0   -6    0   15    0  -20 ¦
```

6-th order discrete Differentiation resp. calculation of the finite 6-th order difference. The rows resp. columns of the matrix Δ^n contain the binomial coefficients, divided by 2^n, in this example the numbers 6!/(k!·(6 - k)!·2^6) = Q0(6, 2k-6).

7 Special differences

horizontal (along localization):

$$Q0(n,k+2) - Q0(n,k) = 2\,\mathrm{QDP}(1,n+1,k+1,\tfrac{1}{2}) = 2\,Q1(n+1,k+1)$$
$$= \frac{-2(k+1)}{n+1}\,Q0(n+1,k+1)$$

vertical (along time):

$$Q0(n+2,k) - Q0(n,k) = \frac{k^2 - n - 2}{(n+2)(n+1)}\, Q0(n+2,k)$$

Correspondence in the middle:

$$Q1(n+1,1) = Q0(n+2,0) - Q0(n,0) = \frac{1}{2}(Q0(n,2) - Q0(n,0))$$

7.1 Schrödinger discretely

$$Q0(n,k-2) - 2\,Q0(n,k) + Q0(n,k+2) \;= 4\,\mathrm{QDP}(2, n+2, k, \frac{1}{2})$$
$$= \;4\,(Q0(n+2,k) - Q0(n,k))$$
$$Q1(n,k-2) - 2\,Q1(n,k) + Q1(n,k+2) = 4\,\mathrm{QDP}(3, n+2, k, \frac{1}{2})$$
$$= 4\,(Q1(n+2,k) - Q1(n,k))$$

Remark: Relation of the discrete second derivative of Q0(n-2,k) to Q0(n,k):

$$Q0(n-2,k-2) - 2\,Q0(n-2,k) + Q0(n-2,k+2) \;= Q0(n,k)\frac{4(k^2-n)}{n(n-1)}$$

8 Scalar products

8.1 horizontal

$$\sum_{k=-m/2}^{m/2} Q0(m,2k)\,Q0(n,2k+j) \;=\; Q0(m+n,j)$$

$$\sum_{k=-n/2}^{n/2} Q0(n,2k)^2 \;=\; Q0(2n,0)$$

$$\sum_{k=-m/2}^{m/2} Q1(m,2k)\,Q1(n,2k) \;=\; Q1(m+n-1,-1) \;=\; -Q2Z(m+n)$$

$$2\sum_{k=-n/2}^{n/2} (2k)^2\;Q0(n,2k)^2 - 0.5\sum_{k=-(n+1)/2}^{(n-1)/2} Q0(n-1,2k)^2 = n\,Q0(2n-2,0)$$

8.2 vertical

$\sum_{n=1}^{2j-1}\sum_{k=-n/2}^{n/2} Q1(n,2k)\,Q1(2j-n,2k) = -(2j-1)\,Q2Z(2j) = Q0Z(2j)$ {skahove}

$\sum_{k=0}^{n} Q2Z(2k)\,Q2Z(2n-2k) = \left(\sum_{k=1}^{n} Q2Z(2k)\,Q2Z(2n-2k)\right) + Q2Z(2n) \;=\; 0$.

$\sum_{k=0}^{n}\left(\sum_{j=0}^{k} Q2Z(2j)\right)\left(\sum_{j=0}^{n-k} Q2Z(2j)\right) \;=\; \sum_{k=0}^{n} Q0Z(2k)\,Q0Z(2n-2k) \;=\; 1$.

8.3 Orthogonality after multiple discrete differentiation (analogously to Hermite polynomials)

{HermPolDiscrete}

Let be d,l ≥ n.

We define the weighted scalar product QSP by

$$\mathrm{QSP}(d,l,n,p) := \sum_{k=-n/2}^{n/2} \frac{1}{Q0P(n,2k,p)}\;\mathrm{QDP}(d,n,2k,p)\;\mathrm{QDP}(l,n,2k,p).$$

Then for d≠l is valid: $QSP(d,l,n,p)=0$ (i.e. orthogonality) and otherwise

$$QSP(d,d,n,p)=\frac{1}{2^n\big(4p(1-p)\big)^d Q0(n,n-2d)}=\frac{1}{p^{2d-n}4^d Q0P(n,n-2d,p)}$$

particularly

$$QSP\left(d,d,n,\frac{1}{2}\right)=\frac{1}{2^n Q0(n,n-2d)}$$

The denominator in the last expression corresponds to the number of the way possibilities from point (0,0) to point (n,n-2d) in the Q0-triangle.

9 Sums

$$Q0Z(2n)=\sum_{m=0}^{n} Q2Z(2m)=1+\sum_{m=1}^{n} Q2Z(2m)=\ 2\sum_{k=-n}^{-1} Q1(2n,2k)$$

10 Moments

10.1 vertical

$$\frac{\sum_{m=0}^{n} Q0Z(2m)}{(2n+1)\ Q0Z(2n)}=\frac{\sum_{m=0}^{n-1} Q0Z(2m)}{2n\ Q0Z(2n)}=\frac{-Q0Z(2n)}{(2n-1)\ Q2Z(2n)}=\frac{3\sum_{m=0}^{n} 2m\ Q0Z(2m)}{2n(2n+1)\ Q0Z(2n)}=\ 1$$

10.2 horizontal

$$-\sum_{k=-n}^{n} Q1(2n\,,2k\,)\,2k\ =\ -\sum_{k=-n}^{n+1} Q1(2n+1\,,2k-1)\,(2k-1)\ =\ 1$$

$$\sum_{k=0}^{n} 2k\ Q0(2n,2k)\ =\ n\ Q0Z(2n)$$

$$2\sum_{k=0}^{n/2}(2k)^2\ \ Q0(n,2k)=\ 4\sum_{k=0}^{n/4}(4k)^2\ \ Q0(n,4k)\ =\ n$$

$$\sum_{k=-n}^{0}(2k)^2\ \ Q1(2n,2k)=\ 2n\ Q0Z(n)$$

$$\sum_{k=-n}^{0}(2k)^3\ \ Q1(2n,2k)=\ 1-3\,n$$

10.2.1 relative to the border

$$\sum_{k=-n/2}^{n/2}(2k+n)^2\ Q0(n,2k)=\ n(n+1)$$

$$\sum_{k=-n/2}^{n/2}(2k-n)^2\ Q1(n,2k)=\ 2n$$

11 Sums and moments for variable p

This chapter contains some elementary formulae for variably p (and n>0).

The first finite difference (discrete derivative) of Q0P is

$$Q1P(n,k,p) := \frac{Q0P(n-1,k+1,p) - Q0P(n-1,k-1,p)}{2}$$

11.1 Sums

$$\sum_{k=-n/2}^{n/2} Q0P(n,2k,p) = 1$$

$$\sum_{k=-n/2}^{n/2} Q1P(n,2k,p) = 0$$

11.2 Deviation relative to the border

In the border the probabilities p and 1-p are very different. With p->0 also v->0 (low temperature).

$$\sum_{k=-n/2}^{n/2} (n+2k)\, Q0P(n,2k,p) = 2\mathrm{np}$$

$$\sum_{k=-n/2}^{n/2} (-n-2k)\, Q1P(n,2k,p) = 1$$

11.3 Deviation relative to the origin

In the center the probabilities p and 1-p are nearly equal, i.e. p-> 1/2 and with that v-> C (the borderline case p=1/2 resp. v=C is represented by Q0 and Q1).

$$\sum_{k=-n/2}^{n/2} (2k)\, Q0P(n,2k,p) = 2\mathrm{np} - n = n(2p-1)$$

$$\sum_{k=-n/2}^{n/2} (-2k)\, Q1P(n,2k,p) = 1$$

12 Analytic representations

Let be

$$Q0E(n,k) := \sqrt{\frac{2}{\pi\, n}}\, e^{-k^2/\,(2n)}$$

$$Q1E(n,k) := -\frac{k}{n}\, Q0E(n,k)$$

i.e.

$$Q1E(n,k) = \frac{\partial}{\partial k}\, Q0E(n,k)$$

then is valid for every sequence (k_n) with (k_n)^3/n^2→0 für n→∞ (p. 80 [likr])

$$\lim_{n\to\infty}\left[\frac{Q0E(n,k_n)}{Q0(n,k_n)}, \frac{Q1E(n,k_n)}{Q1(n,k_n)}\right] = [1,1]$$

Again tested approximation: For |k/n| << 1 (for example up to n=100 for |k/n| < 0.2) the function Q0(n,k)=Q0P(n,k,0.5) is related to the normal distribution:

$$Q0(n,k)/Q\,0E(n,k) = Q0(n,k)\Big/\left(\left(\sqrt{\frac{2}{n\pi}}\right)e^{\left(-\frac{k^2}{2n}\right)}\right) \to 1$$

for larger |k/n| this quotient is first decreasing, then invalid.

12.1 Schrödinger analytically

$$\frac{\partial^2}{\partial k^2}\,Q0E(n,k) \;=\; 2\,\frac{\partial}{\partial n}\,Q0E\,(n,k) \;=\; \frac{k^2-n}{n^2}\,Q0E\,(n,k)$$

$$\frac{\partial^2}{\partial k^2}\,Q1E(n,k) \;=\; 2\,\frac{\partial}{\partial n}\,Q1E\,(n,k) \;=\; \frac{k^2-3n}{n^2}\,Q1E\,(n,k)$$

$$\left[\frac{\partial}{\partial k}\right]^{2j} Q0E(n,k) \;=\; 2^j\left[\frac{\partial}{\partial n}\right]^{j} Q0E(n,k)$$

$$\left[\frac{\partial}{\partial k}\right]^{2j} Q1E(n,k) \;=\; 2^j\left[\frac{\partial}{\partial n}\right]^{j} Q1E(n,k)$$

12.2 Behavior for n-> inf; Dirac delta-function

{DiracDeltaFu} It is

$$\int_{-\infty}^{\infty}\frac{Q0E(n,k)}{2}\,\mathrm{dk} = 1$$

The behavior for n→∞ can be illustrated by a n proportional scale fitting, i.e. by a horizontal compression and a vertical stretching by respectively the factor n. This doesn't touch the value of the integral:

$$\int_{-\infty}^{\infty}\frac{n\,Q0E(n,\mathrm{nk})}{2}\,\mathrm{dk} = \int_{-\infty}^{\infty}\sqrt{\frac{n}{2\pi}}\,e^{-k^2 n/2}\,\mathrm{dk} \;=1$$

For n→∞ therefore the function f(x) = n Q0E(n, nx) / 2 converges towards the Dirac delta-function.

1.2 Multiple differentiation and Hermite polynomials

{HermPol} The Hermite polynomials $H_n(x)$ are (except sign) special cases of pre-factors resulting from multiple differentiation:

$$H_n(x) = (-1)^n\frac{\left[\frac{\partial}{\partial x}\right]^n Q0E(\frac{1}{2},x)}{Q0E(\frac{1}{2},x)} = (-1)^n\frac{\left[\frac{\partial}{\partial x}\right]^n Q0E(100,10x\sqrt{2})}{Q0E(100,10x\sqrt{2})}$$