

A discrete and finite approach to past proper time

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Abstract

The function $\gamma(x) = \frac{1}{\sqrt{1-x^2}}$ plays an important role in mathematical physics, e.g. as factor for relativistic time dilation in case of $x = \beta$ with $\beta = \frac{v}{c}$ or $\beta = \frac{pc}{E}$. Due to former considerations [15] it is reasonable to study the power series expansion of $\gamma(x)$. Here its relationship to the binomial distribution is shown, especially the fact, that the summands of the power series correspond to the return probabilities to the starting point (local coordinates, configuration or state) of a Bernoulli random walk. So $\gamma(x)$ and with that also proper time is proportional to the sum of the return probabilities. In case of $x = 1$ or $v = c$ the random walk is symmetric. Random walks with absorbing barriers are introduced in the appendix. Here essentially the basic mathematical facts are shown and references are given, most interpretation is left to the reader.

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1 Introduction

It has been shown [15], that the (measurable) result data vector of a physical experiment (with finite duration) can be calculated from the (measurable) initial data vector by combining a *finite* number of basic arithmetic operations (addition, subtraction, multiplication, division). This doesn't contradict the fact, that many analytical functions with *infinite* power series expansions can successfully predict (approximative) experimental results: They are only successful in case of convergence, i.e. in case of convergence of the partial sum sequence of the corresponding power series expansion. These partial

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sums can be calculated by finitely many basic arithmetical operations and the result is arbitrarily near to the one of the function. So there always can be an exact partial sum and an approximative function result without chance for experimental distinction. However, the study of partial sums is one possibility to learn more about the nature of the underlying (finite) physical process - even in case of missing convergence.

Here we study the function

$$\gamma :]-1, 1[\rightarrow \mathbb{R}, \quad \gamma(x) = \frac{1}{\sqrt{1-x^2}} \quad (1)$$

which is frequently used in mathematical physics, e.g. as factor for relativistic time dilation in case of $x = \beta$ with $\beta = \frac{v}{c}$ or $\beta = \frac{pc}{E}$. We investigate the power series representation of $\gamma(x)$ and show its (simple?) relationship to the binomial distribution, which plays an important role in nature, often in not obvious way [18]-[48]¹. In the appendix we look also at $\frac{1}{\gamma(x)}$.

2 The connection of proper time and return probabilities

2.1 The binomial series

In case of $\alpha \in \mathbb{Z}^* = \{j \in \mathbb{Z} | j \geq 0\}$ the function

$$\hat{f}_\alpha : \mathbb{C} \rightarrow \mathbb{C} \quad \hat{f}_\alpha(z) = (1+z)^\alpha$$

has a finite power series expansion of the form

$$\hat{f}_\alpha(z) = 1 + \binom{\alpha}{1} z^1 + \binom{\alpha}{2} z^2 + \dots + \binom{\alpha}{\alpha} z^\alpha = \sum_{l=0}^{\alpha} \binom{\alpha}{l} z^l,$$

in which $\binom{\alpha}{l}$ are the *binomial coefficients*. These are defined by

$$\binom{\alpha}{0} = 1 \quad \text{and} \quad \binom{\alpha}{l} = \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-l+1)}{l!} \quad \text{for } l \in \mathbb{Z}^* \setminus \{0\}. \quad (2)$$

¹Recall the close connection between relativistic mass increase and time dilation, especially when reading [22], in which concrete physical relevance of finite partial sums (of the power series expansion of $\frac{1}{\sqrt{1-x}}$) is shown.

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In case of $\alpha \in \mathbb{C} \setminus \mathbb{Z}^*$ and $|z| < 1$ we can develop the function $\hat{f}_\alpha(z) = (1+z)^\alpha$ into a convergent MacLaurin series [5] [10] [11] [12]. If $f_\alpha(z)$ denotes the principal value of $\hat{f}_\alpha(z)$, which is equal to one at $z = 0$, we obtain

$$f_\alpha(0) = 1, \quad f'_\alpha(0) = \alpha, \quad f''_\alpha(0) = \alpha(\alpha - 1), \quad \dots, \quad f_\alpha^{(l)} = \binom{\alpha}{l} l!$$

from which follows the representation of $f_\alpha(z)$ as *binomial series*

$$f_\alpha(z) = \sum_{l=0}^{\infty} \binom{\alpha}{l} z^l. \quad (3)$$

2.2 The power series of $\gamma(x) = \frac{1}{\sqrt{1-x^2}}$

Because of $\frac{1}{\sqrt{1+z}} = f_{-1/2}(z)$ we get with (2) and (3)

$$\begin{aligned} \frac{1}{\sqrt{1+z}} &= \sum_{l=0}^{\infty} \binom{-\frac{1}{2}}{l} z^l \\ &= 1 + \frac{-\frac{1}{2}}{1} z^1 + \frac{-\frac{1}{2} \cdot -\frac{3}{2}}{1 \cdot 2} z^2 + \frac{-\frac{1}{2} \cdot -\frac{3}{2} \cdot -\frac{5}{2}}{1 \cdot 2 \cdot 3} z^3 + \frac{-\frac{1}{2} \cdot -\frac{3}{2} \cdot -\frac{5}{2} \cdot -\frac{7}{2}}{1 \cdot 2 \cdot 3 \cdot 4} z^4 + \dots \\ &= 1 - \frac{1}{2^1 \cdot 1!} z^1 + \frac{1 \cdot 3}{2^2 \cdot 2!} z^2 - \frac{1 \cdot 3 \cdot 5}{2^3 \cdot 3!} z^3 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2^4 \cdot 4!} z^4 - \dots \\ &= 1 - \frac{2!}{(2^1 \cdot 1!)^2} z^1 + \frac{4!}{(2^2 \cdot 2!)^2} z^2 - \frac{6!}{(2^3 \cdot 3!)^2} z^3 + \frac{8!}{(2^4 \cdot 4!)^2} z^4 - \dots \\ &= \sum_{l=0}^{\infty} (-1)^l \frac{(2l)!}{(2^l \cdot l!)^2} z^l \\ &= \sum_{l=0}^{\infty} \binom{2l}{l} \left(\frac{-z}{4}\right)^l \end{aligned} \quad (4)$$

and after substitution of z by $-x^2$

$$\gamma(x) = \frac{1}{\sqrt{1-x^2}} = \sum_{l=0}^{\infty} \binom{2l}{l} \left(\frac{x}{2}\right)^{2l}. \quad (5)$$

2.3 Bernoulli random walk

A Bernoulli random walk is a stochastic process generated by a sequence of Bernoulli trials² [2][16]. It can be interpreted as a model for the movement

²Independent trials, each one of which can have only two results, e.g. "positive" (with probability p) or "negative" (with probability $1 - p$).

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of a particle in a one-dimensional discrete state space and may be described in the following terms: The particle moves "randomly" along a line over a lattice of equidistant points ("states"), which are indexed by an integer coordinate k . With every trial the particle makes a step from point k to point $k + 1$ with given probability p ("positive direction") or a step from point k to point $k - 1$ with probability $1 - p$ ("negative direction").

For $n \in \{1, 2, 3, \dots\}$ let's denote by $Q0P(n, k, p)$ the probability, that the particle is at point k after the n -th step and by $Q0P(0, k, p)$ this probability before the first step. We assume start of movement at $k = 0$, so $Q0P(0, 0, p) = 1$ and $Q0P(0, k, p) = 0$ for $k \neq 0$ and furthermore

$$Q0P(n + 1, k, p) = p Q0P(n, k - 1, p) + (1 - p) Q0P(n, k + 1, p) . \quad (6)$$

When making n trials, point k is only within reach, if $n - k$ and $n + k$ are non-negative even numbers. We will presuppose this subsequently. There are exactly $\binom{n}{\frac{n+k}{2}}$ ways with $\frac{n+k}{2}$ steps in positive and $\frac{n-k}{2}$ steps in negative direction, which lead into point k after the n -th step. They respectively have the probability $(1-p)^{(n-k)/2} p^{(n+k)/2}$. So the chaining of these Bernoulli trials results into the binomial distribution

$$Q0P(n, k, p) = \binom{n}{\frac{n+k}{2}} p^{(n+k)/2} (1-p)^{(n-k)/2} . \quad (7)$$

We now look at the probabilities of return to the starting point. Because the movement started at $k = 0$ these correspond to

$$Q0P(2n, 0, p) = \binom{2n}{n} (1-p)^n p^n ,$$

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i.e. $Q0P(2n, 0, p)$ is the *return probability* after the $2n$ -th step³. Substitution of p by $\frac{1-\sqrt{1-x^2}}{2}$ or $\frac{1+\sqrt{1-x^2}}{2}$ yields

$$\begin{aligned}
 & Q0P\left(2n, 0, \frac{1 - \sqrt{1 - x^2}}{2}\right) \\
 = & Q0P\left(2n, 0, \frac{1 + \sqrt{1 - x^2}}{2}\right) \\
 = & \binom{2n}{n} \left(\frac{1 - \sqrt{1 - x^2}}{2} \frac{1 + \sqrt{1 - x^2}}{2}\right)^n \\
 = & \binom{2n}{n} \left(\frac{x^2}{4}\right)^n = \binom{2n}{n} \left(\frac{x}{2}\right)^{2n}
 \end{aligned} \tag{8}$$

and with (5) we obtain

$$\begin{aligned}
 \gamma(x) = \frac{1}{\sqrt{1-x^2}} &= \sum_{n=0}^{\infty} Q0P\left(2n, 0, \frac{1 - \sqrt{1 - x^2}}{2}\right) \\
 &= \sum_{n=0}^{\infty} Q0P\left(2n, 0, \frac{1 + \sqrt{1 - x^2}}{2}\right).
 \end{aligned} \tag{9}$$

Note, that the condition

$$p \in \left\{ \frac{1 - \sqrt{1 - x^2}}{2}, \frac{1 + \sqrt{1 - x^2}}{2} \right\}$$

is equivalent to

$$4p(1-p) = x^2. \tag{10}$$

Before we continue, we should remember, that the function $\gamma(x)$ cannot have an exact equivalent in physical (past) reality, because the sum on the right side of (9) is not finite. Furthermore the values $Q0P(2n, 0, p)$ are probabilities, and every expectation value calculated from probabilities has only an average, approximative meaning. Therefore we presuppose, that the random walk contains a sufficiently large number of steps, so that there can be an equivalent to a finite partial sum of the right side of (9) sufficiently close to $\gamma(x)$, that the reliability of the expectation value calculated from it is so great, that the difference between the individual (discrete⁴) measurement result and the calculated value isn't significant. With that we can summarize:

³Return is only possible after an even number of steps.

⁴Due to [15] the set of all possible measurement results is finite and so of course also discrete.

2.4 Theorem: Proper time proportional to the sum of return probabilities

Let $\gamma(x) = \frac{1}{\sqrt{1-x^2}}$ represent the (approximative) time dilation factor of reference system A relative⁵ to reference system B. Then proper time of A relative to B is (approximatively) proportional to the sum of the return probabilities to the starting point of a Bernoulli random walk, in which each step is directed from point⁶ k to $k + 1$ with probability p , from point k to $k - 1$ with probability $1 - p$ and $4p(1 - p) = x^2$.

2.5 Case $x = 1$ resp. $v = c$

In many physical situations $x = 1$, especially if $x = \frac{v}{c}$ and $v = c$ is the velocity of light resp. photons⁷. So the case $x = 1$ is extremely frequent. Why?

The above consideration (10) shows, that $x = 1$ corresponds to $p = 1 - p = \frac{1}{2}$, i.e. the probabilities p and $1 - p$ of positive and negative step direction are equal. Now the reason of $x = 1$ resp. $v = c$ for photons becomes clear: At this both directions have the same chance. Nature *a priori* makes no preferences.

2.5.1 Symmetric random walk

In case of $x = 1$ because of $p = 1 - p = \frac{1}{2}$ the random walk is symmetric. The accompanying probabilities are

$$Q0(n, k) := Q0P(n, k, \frac{1}{2}) = \binom{n}{\frac{n+k}{2}} \left(\frac{1}{2}\right)^n. \quad (11)$$

⁵We can assume $x = \frac{v}{c}$ if B is moving with velocity v relative to A and space-time is flat.

⁶Every point can represent a state in a one-dimensional discrete state space and k the integer index to it. Reversal of the order of the index is possible and has the same effect as exchange of the probabilities p and $1 - p$.

⁷Because $v = c$ is the maximal speed of information transport, this case is also important from information theoretical point of view.

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n	$k \rightarrow$	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
\downarrow														
0								<u>1</u>						$.2^0$
1							1		1					$.2^{-1}$
2						1		<u>2</u>		1				$.2^{-2}$
3					1		3		3		1			$.2^{-3}$
4				1		4		<u>6</u>		4		1		$.2^{-4}$
5			1		5		10		10		5		1	$.2^{-5}$
6		1		6		15		<u>20</u>		15		6		$.2^{-6}$
...														

Table 1: The first values of $Q0(n, k)$. The return probabilities are underlined. The representation is chosen in a way that the well known Pascal triangle gets visible. Also the small modification in comparison with usual tables of binomial coefficients gets clear: The definition of $Q0(n, k)$ takes into account the symmetry. The underlined return probabilities are located in the symmetry center and have all the same index $k = 0$.

2.5.2 Finite random walk - finite partial sum $\gamma_{2n}(x)$ of $\gamma(x)$

We now look again to $\gamma(x)$. In case $x = 1$ the series (9) doesn't converge, i.e. the infinite sum has not even an approximative result. But anyway we know that an infinite sum cannot have an equivalent in physical reality. So it's only consequent to consider finite partial sums

$$\gamma_{2n}(x) := \sum_{m=0}^n Q0P \left(2m, 0, \frac{1 + \sqrt{1 - x^2}}{2} \right)$$

of (9). $\gamma_\infty(x) = \gamma(x)$, additionally for every (finite) integer n now also

$$\gamma_{2n}(1) = \gamma_n(-1) = \sum_{m=0}^n Q0P \left(2m, 0, \frac{1}{2} \right) = \sum_{m=0}^n Q0(2m, 0) \quad (12)$$

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exists. It is not difficult to find a closed form for it. From

$$\begin{aligned}
 (2n-1)Q0(2n-2, 0) + Q0(2n, 0) &= (2n-1) \frac{(2n-2)!}{2^{2n-2} (n-1)!^2} + \frac{(2n)!}{2^{2n} n!^2} \\
 &= \frac{2n (2n)!}{(2n)^2 2^{2n-2} (n-1)!^2} + \frac{(2n)!}{2^{2n} n!^2} = \frac{2n (2n)!}{n^2 2^{2n} (n-1)!^2} + \frac{(2n)!}{2^{2n} n!^2} \\
 &= \frac{2n (2n)!}{2^{2n} n!^2} + \frac{(2n)!}{2^{2n} n!^2} = (2n+1) \frac{(2n)!}{2^{2n} n!^2} \\
 &= (2n+1)Q0(2n, 0)
 \end{aligned}$$

follows by induction

$$\sum_{m=0}^n Q0(2m, 0) = (2n+1) Q0(2n, 0)$$

and with (12)

$$\begin{aligned}
 \gamma_{2n}(1) &= (2n+1) Q0(2n, 0) = (2n+1) \binom{2n}{n} \left(\frac{1}{2}\right)^{2n} \\
 &= (2n+1) \frac{(2n)!}{2^{2n} n!^2} \tag{13}
 \end{aligned}$$

In case of large n we can use the Stirling formula $n! \approx n^n e^{-n} \sqrt{2\pi n}$ and obtain $Q0(2n, 0) \approx \frac{1}{\sqrt{\pi n}}$ and $\gamma_{2n}(1) \approx \sqrt{\frac{4n}{\pi}}$. So we have gotten a closed form for the sum (12) of the return probabilities (the sum of the underlined values in table 1 on page 8). The results are finite even in case of $x = 1$ (or $v = c$), because we assumed an only finite number $2n$ of steps. Obviously this assumption is adequate for all natural processes with finite duration.

Comment

The model of a one-dimensional random walk has only limited validity. Extensive considerations should take into account interactions between different reference systems and changes of the own reference system. Up to now we don't know enough about the exact ways of information between different reference systems⁸ and about the long-term relation of their proper time.

⁸Example: Squared values like $\gamma_{2n}(1)^2 \approx \frac{4n}{\pi}$ or $\zeta_{2n}(1)^2 \approx \frac{1}{\pi n}$ (cf. (23) in the appendix) can appear because of *bidirectional* information exchange during observation. The familiar

Appendix

Further research is necessary, also combinatorial and graph theoretical research. The appendix demonstrates an example for possible connections of multiple random walks.

3 Appendix

We now introduce absorbing barriers, which are drains and can be sources of new random walks with steps in another orthogonal direction. Then we show, that in case of an absorbing barrier in the origin after start of the walk (and otherwise under the same the basic conditions as in theorem 2.4) the probability of non-absorption is equivalent to $1/\gamma(x) = \sqrt{1-x^2}$. At last we investigate finite symmetric random walks with absorbing barrier.

3.1 Absorbing barriers

A Bernoulli random walk can have absorbing barriers. If there is an absorbing barrier at point a and the walking particle reaches it, the particle is absorbed. So point a is only a drain, but no (direct) source for further walks within the same dimension⁹. We can get the resulting probability distribution by subtraction of a "shifted" distribution from (7): Let's assume an absorbing barrier at $a > 0$. We define

$$P_a(n, k, p) := Q0P(n, k, p) - \left(\frac{p}{1-p}\right)^a Q0P(n, k - 2a, p) \quad (14)$$

from which follows

$$P_a(n+1, k, p) = p P_a(n, k-1, p) + (1-p) P_a(n, k+1, p), \quad (15)$$

i.e. the inductive law (6) of a Bernoulli random walk holds. Additionally the boundary condition $P_a(n, a, p) = 0$ is fulfilled, so that point a is only drain, but not source¹⁰. Therefore $P_a(n, k, p)$ represents for all values $-n \leq k \leq a$ within reach the probability, that the particle passes point k and continues

macroscopic geometrical appearance isn't a primary thing, it's only a consequence of a discrete law [15]. The above considerations suggest an information theoretical approach to this law.

⁹It can be source of a walk in another dimension.

¹⁰In literature at point a often the sum of absorption probabilities is listened. Here this special treatment is not done, so that law (15) is valid.

Appendix

moving. For the particle starting at $k < a$ the points $k > a$ are not within reach¹¹.

3.1.1 Random walk with delayed absorbing barrier at $k = 0$

The starting coordinate $k = 0$ plays a special role and it is reasonable to assume an absorbing barrier there, also because of symmetry. But if this barrier is active from the beginning on, the particle is absorbed at once so that the walk cannot begin and (14) has the meaningless result $P_0(n, k, p) = Q0P(n, k, p) - Q0P(n, k, p) = 0$. However, if there is absorption at $k = 0$ *after* the walk already has started, we get a distribution which is worth further consideration. So let's assume a *delayed* absorbing barrier at $k = 0$ which is activated *after* completion of the first step of the walk. The resulting probability distribution is given by the absolute values of

$$Q1P(n, k, p) := (1 - p) Q0P(n - 1, k + 1, p) - p Q0P(n - 1, k - 1, p) \quad (16)$$

which is a modification¹² of (14). $Q1P$ fulfills the boundary conditions

$$|Q1P(0, 0, p)| = 1, \quad Q1P(2n, 0, p) = 0 \text{ for } n \geq 1$$

and the same inductive law as $Q0P$ in (6). For $n \geq 1$ a more compact form of $Q1P(n, k, p)$ is

$$Q1P(n, k, p) = \frac{-k}{n} Q0P(n, k, p) \quad (17)$$

because of

¹¹ $P_a(n, k, p)$ is negative there. In case of a simultaneous walk of two particles with starting points 0 and $2a$, in which the particle starting at $2a$ is the annihilating counterpart of the other starting at 0, for $k > a$ the absolute value $|P_a(n, k, p)|$ can be interpreted as probability, that the annihilating counterpart passes point k , if both particles make simultaneously steps in opposite directions. If this is not guaranteed, there is a chance, that a particle passes the barrier (like in the tunnel effect).

¹² $Q1P(n, k, p) = (1 - p) P_1(n - 1, k + 1, p)$; an absorbing barrier at $k = 1$ is within reach and therefore active only from the second step on. $Q1P(n, k, p)$ is "centered" in this barrier.

$$\begin{aligned}
Q1P(n, k, p) &= (1 - p) Q0P(n - 1, k + 1, p) - p Q0P(n - 1, k - 1, p) \\
&= \frac{p^{(n+k)/2} (1 - p)^{(n-k)/2} (n - 1)!}{\left(\frac{n+k}{2}\right)! \left(\frac{n-k}{2} - 1\right)!} - \frac{p^{(n+k)/2} (1 - p)^{(n-k)/2} (n - 1)!}{\left(\frac{n+k}{2} - 1\right)! \left(\frac{n-k}{2}\right)!} \\
&= \left(\frac{n - k}{2n}\right) \frac{p^{(n+k)/2} (1 - p)^{(n-k)/2} n!}{\left(\frac{n+k}{2}\right)! \left(\frac{n-k}{2}\right)!} - \left(\frac{n + k}{2n}\right) \frac{p^{(n+k)/2} (1 - p)^{(n-k)/2} n!}{\left(\frac{n+k}{2}\right)! \left(\frac{n-k}{2}\right)!} \\
&= \left(\frac{-k}{n}\right) \frac{p^{(n+k)/2} (1 - p)^{(n-k)/2} n!}{\left(\frac{n+k}{2}\right)! \left(\frac{n-k}{2}\right)!} = \frac{-k}{n} Q0P(n, k, p) .
\end{aligned}$$

3.1.2 Past differences

Equation (16) has similarity to a finite difference along k . It represents the probability difference of the two ways coming from past. Therefore we shall call the accompanying operator *past difference* and use the symbol $\hat{\Delta}$ for it. If ψ is a function of the variables n, k, p , defined at least at $(n - 1, k + 1, p)$ and $(n - 1, k - 1, p)$, its past difference is

$$\hat{\Delta}\psi(n, k, p) = (1 - p) \psi(n - 1, k + 1, p) - p \psi(n - 1, k - 1, p) . \quad (18)$$

Similarly to usual finite differences we can form higher-order past differences, for example the second-order past difference

$$\begin{aligned}
Q2P(n, k, p) &:= \hat{\Delta}^2 Q0P(n, k, p) = \hat{\Delta}\hat{\Delta}Q0P(n, k, p) \\
&= (1 - p) \hat{\Delta}Q0P(n - 1, k + 1, p) - p \hat{\Delta}Q0P(n - 1, k - 1, p) \\
&= (1 - p)^2 Q0P(n - 2, k + 2, p) + p^2 Q0P(n - 2, k - 2, p) \\
&\quad - 2p(1 - p) Q0P(n - 2, k, p) .
\end{aligned} \quad (19)$$

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For $n \geq 2$ we obtain

$$\begin{aligned}
& \hat{\Delta}^2 Q0P(n, k, p) = \hat{\Delta} \hat{\Delta} Q0P(n, k, p) = \hat{\Delta} Q1P(n, k, p) \\
& = \hat{\Delta} \left(\frac{-k}{n} Q0P(n, k, p) \right) \\
& = (1-p) \left(\frac{-k-1}{n-1} \right) Q0P(n-1, k+1, p) - p \left(\frac{1-k}{n-1} \right) Q0P(n-1, k-1, p) \\
& = \left(\frac{-k-1}{n-1} \right) ((1-p) Q0P(n-1, k+1, p) - p Q0P(n-1, k-1, p)) \\
& \quad - p \left(\frac{2}{n-1} \right) Q0P(n-1, k-1, p) \\
& = \left(\frac{-k-1}{n-1} \right) Q1P(n, k, p) - \left(\frac{2}{n-1} \right) \left(\frac{n+k}{2n} \right) Q0P(n, k, p) \\
& = \left(\frac{k(k+1)}{n(n-1)} \right) Q0P(n, k, p) - \left(\frac{n+k}{n(n-1)} \right) Q0P(n, k, p) \\
& = \frac{k^2 - n}{n(n-1)} Q0P(n, k, p) .
\end{aligned}$$

The *central* second-order past differences

$$Q2P(2n, 0, p) = \frac{-1}{2n-1} Q0P(2n, 0, p) = \frac{-1}{2n-1} \binom{2n}{n} (1-p)^n p^n \quad (20)$$

have a special meaning: Because of

$$\begin{aligned}
|Q2P(2n, 0, p)| &= |\hat{\Delta} Q1P(2n, 0, p)| \\
&= |(1-p) Q1P(2n-1, 1, p) - p Q1P(2n-1, -1, p)| \\
&= (1-p) |Q1P(2n-1, 1, p)| + p |Q1P(2n-1, -1, p)|
\end{aligned}$$

for $n \geq 1$ the absolute values

$$|Q2P(2n, 0, p)| = \frac{Q0P(2n, 0, p)}{2n-1} \quad (21)$$

correspond to the probability of absorption after the $2n$ -th step of the random walk specified in chapter 3.1.1.

It is worth¹³ mentioning that the second order past difference (along k) is equivalent to a weighted first order difference along n :

$$Q2P(n, k, p) = Q0P(n, k, p) - 4p(1-p) Q0P(n-2, k, p) . \quad (22)$$

¹³In important physical equations (e.g. Schrödinger equation) the second derivative along location is related to the first derivative along time.

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This follows from (19) and

$$Q0P(n, k, p) = (1-p)^2 Q0P(n-2, k+2, p) + p^2 Q0P(n-2, k-2, p) + 2p(1-p) Q0P(n-2, k, p).$$

3.2 The power series of $1/\gamma(x) = \sqrt{1-x^2}$

Just like in chapter 2.2 we now look at the power series of

$$\zeta : [-1, 1] \rightarrow \mathbb{R}, \quad \zeta(x) = \sqrt{1-x^2};$$

$\zeta(x) = \frac{1}{\gamma(x)}$ for $|x| < 1$ and $\zeta(-1) = \zeta(1) = 0$. $\sqrt{1+z} = f_{1/2}(z)$, so analogously to (4) we get

$$\begin{aligned} \sqrt{1+z} &= \sum_{l=0}^{\infty} \binom{\frac{1}{2}}{l} z^l \\ &= 1 + \frac{\frac{1}{2}}{1} z^1 + \frac{\frac{1}{2} \cdot -\frac{1}{2}}{1 \cdot 2} z^2 + \frac{\frac{1}{2} \cdot -\frac{1}{2} \cdot -\frac{3}{2}}{1 \cdot 2 \cdot 3} z^3 + \frac{\frac{1}{2} \cdot -\frac{1}{2} \cdot -\frac{3}{2} \cdot -\frac{5}{2}}{1 \cdot 2 \cdot 3 \cdot 4} z^4 + \dots \\ &= 1 + \frac{1}{2^1 \cdot 1!} z^1 - \frac{1 \cdot 1}{2^2 \cdot 2!} z^2 + \frac{1 \cdot 1 \cdot 3}{2^3 \cdot 3!} z^3 - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2^4 \cdot 4!} z^4 + \dots \\ &= 1 - \sum_{l=1}^{\infty} \frac{1}{2l-1} \binom{2l}{l} \left(\frac{-z}{4}\right)^l \end{aligned}$$

from which follows

$$\zeta(x) = \sqrt{1-x^2} = 1 - \sum_{l=1}^{\infty} \frac{1}{2l-1} \binom{2l}{l} \left(\frac{x}{2}\right)^{2l}.$$

So in case of $4p(1-p) = x^2$ we obtain with (8), (10) and (20)

$$\begin{aligned} \zeta(x) = \sqrt{1-x^2} &= 1 - \sum_{n=1}^{\infty} \frac{1}{2n-1} Q0P(2n, 0, p) \\ &= 1 + \sum_{n=1}^{\infty} Q2P(2n, 0, p) = 1 - \sum_{n=1}^{\infty} |Q2P(2n, 0, p)|. \end{aligned}$$

Because $|Q2P(2n, 0, p)|$ is the probability of absorption after the $2n$ -th step, $\sum_{n=1}^{\infty} |Q2P(2n, 0, p)|$ is the total probability of absorption. Therefore we conclude:

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3.2.1 Theorem: $\sqrt{1-x^2}$ as probability of non-return (of "escape")

If a particle makes a Bernoulli random walk, in which each step is directed from point k to $k+1$ with probability p , from point k to $k-1$ with probability $1-p$ and $4p(1-p) = x^2$ and the particle is absorbed if it returns to the starting point, the probability of non-absorption (of "escape") is $\zeta(x) = \sqrt{1-x^2}$.

Remark. Also more concrete formulations of this theorem are possible. Due to experimental results we know, that the energy of a photon can be distributed. If E is the energy of the photon, its frequency ν is given by $\nu = \frac{E}{h}$, in which h is Planck's constant ($h \approx 6.626 \cdot 10^{-34} \text{ Js}$). At this a reduction of the photon's frequency is equivalent to a dilation of its time period. So we can state:

3.2.2 Theorem: Energy of a received photon as non-returning (escaping) part of its initial energy

Let $\gamma(x) = \frac{1}{\sqrt{1-x^2}}$ represent the (approximative) time dilation factor of reference system A relative to reference system B as in theorem 2.4. If a photon is emitted in B with energy $E_e = h\nu_e$ and absorbed in A, the maximal absorption energy $E_a = h\nu_a$ in system A is given by $E_a = E_e\sqrt{1-x^2}$. So the quotient¹⁴ $\frac{E_a}{E_e} = \frac{\nu_a}{\nu_e}$ is (approximatively) equivalent to the probability¹⁵, that there is no return to the starting point during a Bernoulli random walk, in which each step is directed from point k to $k+1$ with probability p , from point k to $k-1$ with probability $1-p$ and $4p(1-p) = x^2$.

3.3 Case $x = 1$ resp. $v = c$ with absorbing barrier

3.3.1 Symmetry

In case of $x = 1$ or $v = c$ also the chapter 3.1.1 described random walk with absorbing barrier becomes symmetric, because the (after the first step active) barrier is located in the starting point $k = 0$ and $p = 1 - p = \frac{1}{2}$ with (10). The probability, that after the n -th step point k is reached and the

¹⁴The part of the photon's energy which can escape and arrive in A in comparison to initial energy of the photon

¹⁵The expectation value of the frequency of non-returning (escaping) walks in comparison to the total frequency or total number of walks

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walk continues, is given by the absolute value of

$$Q1(n, k) := Q1P\left(n, k, \frac{1}{2}\right).$$

n	$k \rightarrow$	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
\downarrow														
1							<u>1</u>	-1						$\cdot 2^{-1}$
2						1	0	-1						$\cdot 2^{-2}$
3					1	<u>1</u>	-1	-1						$\cdot 2^{-3}$
4				1	2	0	-2	-1						$\cdot 2^{-4}$
5			1	3	<u>2</u>	-2	-3	-1						$\cdot 2^{-5}$
6		1	4	5	0	-5	-4	-1						$\cdot 2^{-6}$
...														

Table 2: The first values of $Q1(n, k) = Q1(n, k, \frac{1}{2})$. $|Q1(n, k)|$ is the probability, that after the n -th step point k is reached and the walk continues, therefore $Q1(2n, 0) = 0$ in the absorbing barrier. The underlined values $Q1(2n - 1, -1) = |Q1(2n - 1, 1) - Q1(2n - 1, -1)|/2 = |Q2P(2n, 0, \frac{1}{2})|$ are the probabilities of absorption after the $2n$ -th step. It is visible, that the numbers result from addition of two symmetric binomial distributions with opposite sign, one starting at $(n, k) = (1, -1)$, the other starting at $(n, k) = (1, 1)$, so that at $k = 0$ annihilation occurs.

3.3.2 Finite random walk

With 3.2.1 in case of $x = 1$ the probability of absorption (or return to the starting point) is 1 if the number of steps in the walk has no upper limit. Because in physical reality within finite time only a finite number of steps are possible we consider finite partial sums

$$\zeta_{2n}(x) := 1 + \sum_{m=1}^n Q2P\left(2m, 0, \frac{1 + \sqrt{1 - x^2}}{2}\right)$$

of the power series of $\zeta(x)$. Similarly as in chapter 2.5.2 for $\gamma_{2n}(1)$ we can find a closed form for $\zeta_{2n}(1)$. For $n > 0$ we get with (22)

$$Q0P\left(n - 2, 0, \frac{1}{2}\right) + Q2P\left(n, 0, \frac{1}{2}\right) = Q0P\left(n, 0, \frac{1}{2}\right)$$

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so that with $Q0P(0, 0, \frac{1}{2}) = 1$ by induction follows

$$\zeta_{2n}(1) = 1 + \sum_{m=1}^n Q2P\left(2m, 0, \frac{1}{2}\right) = Q0P\left(2n, 0, \frac{1}{2}\right) = \frac{(2n)!}{2^{2n}(n!)^2}.$$

In case of large n we can use the Stirling formula and obtain

$$\zeta_{2n}(1) \approx \frac{1}{\sqrt{\pi n}}. \quad (23)$$

$1 - \zeta_{2n}(1)$ is the probability of absorption in case of $x = 1$ or $p = 1 - p = \frac{1}{2}$ when making at most $2n$ steps. The probability of absorption (21) after the $2n$ -th step is given by the negative second-order past difference (along k)

$$\begin{aligned} -Q2P\left(2n, 0, \frac{1}{2}\right) &= -\hat{\Delta}^2 Q0P\left(2n, 0, \frac{1}{2}\right) = \frac{1}{2n-1} Q0P\left(2n, 0, \frac{1}{2}\right) \\ &\approx \frac{1}{\sqrt{4\pi n^3}} \end{aligned}$$

and because of the Schrödinger equation it is remarkable, that with (22) this is equivalent to the negative (first-order) finite difference along n :

$$-Q2P\left(2n, 0, \frac{1}{2}\right) = Q0P\left(2n-2, 0, \frac{1}{2}\right) - Q0P\left(2n, 0, \frac{1}{2}\right).$$

We have seen, that the in chapter 3.1.2 defined "discrete differentiation" leads to a probability distribution with absorbing barrier. Separation (and distinction) of the ways on both sides of the barrier is connected with this. We should recall, that in physical experiments (e.g. double slit experiment) such separation also is connected with absorption - and emission - of photons at systems with rest mass.

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